MANY LIVES OF KASHIWARA - VERGNE EQUATIONS

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based on joint works with

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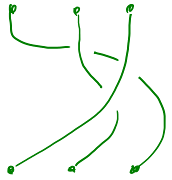
Inspired by

"THE CAMPBELL - HAUSDORFF FORMULA AND INVARIANT HYPERFUNCTIONS"

MASAKI KASHIWARA and
MICHÈLE VERGNE

Inventiones math. 47
(1978) 249 - 272

Braids & Associators



$\Phi\Phi\Phi = \Phi\Phi$

Duflo Theorem

$Z(U\mathfrak{g}) \cong (S\mathfrak{g})^{\mathfrak{g}}$

KZ equation

$d + \sum t_{ij}\, d\log(z_{ij})$

KV I : $\log(e^x e^y) = \dots$

KV II : $\mathrm{Tr}_{\mathfrak{g}}\, \mathrm{ad}_{?} = \mathrm{Tr}_{\mathfrak{g}}\, \mathrm{ad}_{??}$

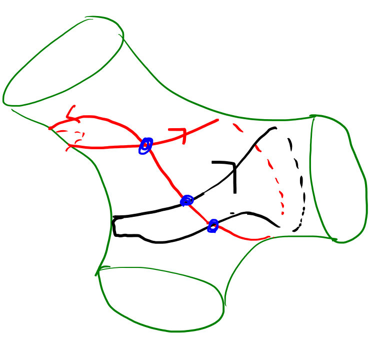
MZVs

$\zeta(n_1, \dots, n_d)$

4D topology

Goldman-Turaev

$[\cdot,\cdot]\,,\ \delta$



BV

$\Delta^2 = 0$

KASHIWARA - VERGE PROBLEM

FIND $F, G \in \mathbb{L}(x,y)$

free Lie, generators x, y

such that

- KV I:

$$(1-e^{-\mathrm{ad}_x})F + (e^{\mathrm{ad}_y}-1)G = x+y-\underbrace{\log(e^y e^x)}_{\text{BCH series}}$$

- KV II:

$\forall\ \mathfrak{g}$ = Lie algebra, $\dim \mathfrak{g} < +\infty$

$$\mathrm{Tr}_{\mathfrak{g}}\left[\mathrm{ad}_x \circ \partial_x F + \mathrm{ad}_y \circ \partial_y G\right] =$$

$$= \frac{1}{2}\mathrm{Tr}_{\mathfrak{g}}\left[\frac{\mathrm{ad}_x}{e^{\mathrm{ad}_x}-1} + \frac{\mathrm{ad}_y}{e^{\mathrm{ad}_y}-1} - \frac{\mathrm{ad}_z}{e^{\mathrm{ad}_z}-1} - 1\right]$$

Here: $z = \log(e^x e^y)$, $\partial_x F(w) = \frac{d}{dt} F(x+tw, y)\Big|_{t=0}$

KV I

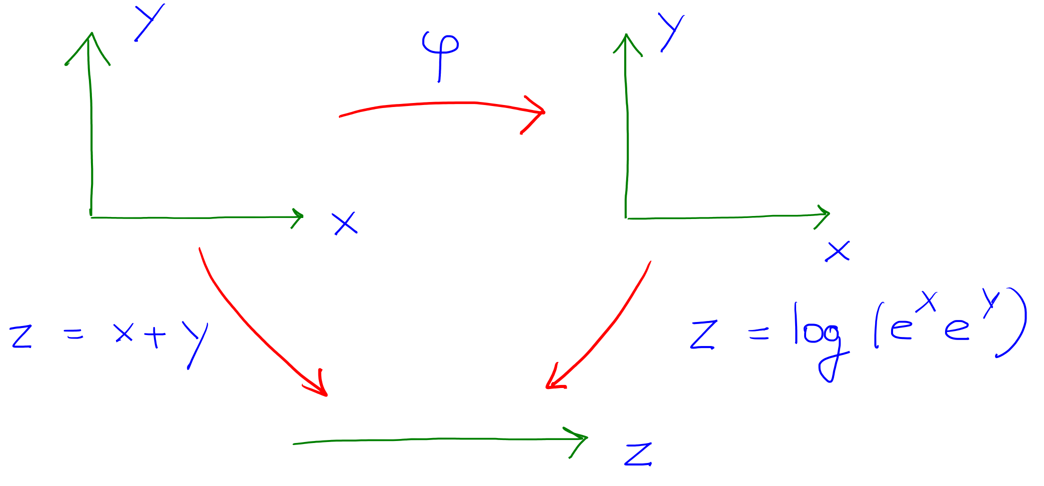
FACT : KV I equation

$$\Updownarrow$$

$\exists\ \varphi \in \mathrm{Aut}(\mathbb{L}(x,y))$ s.t.

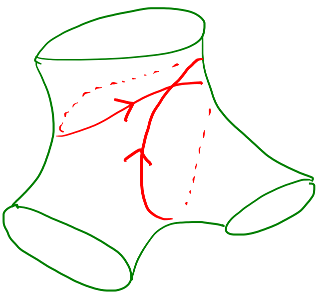
- $\varphi(\log(e^x e^y)) = x + y$
- $\varphi(x) \sim x$, $\varphi(y) \sim y$

Non-commutative change of variables:



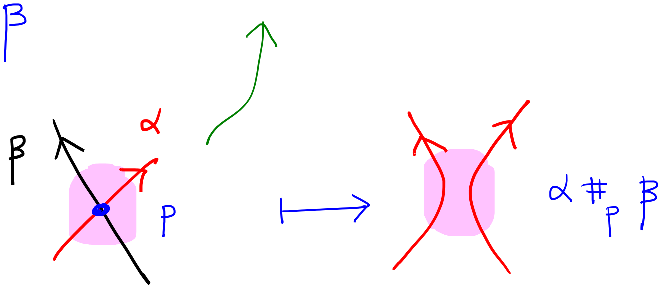
Recall: Goldman bracket

Σ = sphere with 3 holes



$\mathfrak{g}(\Sigma) = \Bbbk\langle$ homotopy classes of oriented loops $\rangle$

$$[\alpha, \beta]_{\text{Goldman}} = \sum_{p \in \alpha \cap \beta} \varepsilon_p \; \alpha \#_p \beta$$



Remark: $\pi_1(\Sigma) \cong \text{Free}(X, Y)$

$$\mathfrak{g}(\Sigma) \cong \text{Cycl}(X, X^{-1}, Y, Y^{-1})$$

Filtrations & associated graded

$$\pi_1(\Sigma) \supset [\pi_1(\Sigma), \pi_1(\Sigma)] \supset \ldots$$

$$\downarrow$$

$$\mathfrak{g}(\Sigma) \supset \mathfrak{g}^{(1)}(\Sigma) \supset \mathfrak{g}^{(2)}(\Sigma) \supset \ldots$$

$$\cong$$

$Cycl(x, y)$ cyclic words in 2 letters

$[\cdot, \cdot]_{gr}$ = the graded version of Goldman bracket

$$\|$$

the Kirillov-Kostant-Souriau bracket

Example:

$$[Cycl(xy), x]_{gr} = xy - yx = [x, y]$$

Theorem $\left[\begin{array}{ll} \Rightarrow & \text{Massuyeau-Turaev} \\ & \text{Kawazumi-Kuno} \\ & \text{Naef} \\ \Leftarrow & \text{AKKN} \end{array}\right]$

(i) KVI $\quad \varphi \in \mathrm{Aut}(\mathbb{L}(x,y))$

$$\varphi(x) \sim x, \quad \varphi(y) \sim y$$

$$\varphi(\log(e^x e^y)) \underset{\sim}{=} x + y$$

$$\Updownarrow$$

(ii) φ induces an isomorphism

$$(\mathfrak{g}(\Sigma), [\cdot,\cdot]_{\text{Goldman}}) \cong (\mathrm{Cycl}(x,y), [\cdot,\cdot]_{gr})$$

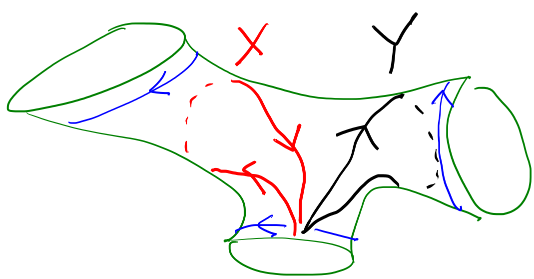
Remark

$$\mathfrak{z}\left(\mathfrak{g}(\Sigma),\ [\cdot\,;\cdot]_{\mathrm{Goldman}}\right)$$

$$\cong$$

$$\Bbbk\left\langle \mathrm{Cycl}(X^n),\ \mathrm{Cycl}(Y^n),\ \mathrm{Cycl}(XY)^n \right\rangle$$

boundary loops



$$\Big\downarrow \varphi$$

$$\mathfrak{z}\left(\mathrm{Cycl}(x,y),\ [\cdot,\cdot]_{gr}\right)$$

$$\cong$$

$$\Bbbk\left\langle \mathrm{cycl}(x^n),\ \mathrm{Cycl}(y^n),\ \mathrm{Cycl}(x+y)^n \right\rangle$$

Key ingredient in the proof $\Leftarrow$

Theorem [AKKN]

- $a, b \in \mathbb{L}(x, y)$
- $\mathrm{cycl}(a^k) = \mathrm{cycl}(b^k) \quad \forall k$
- $a = \alpha x + \beta y + \ldots$

  $(\alpha, \beta) \neq 0$

$\Rightarrow \quad a \sim b$

" no Jordan blocks in non-commutative calculus "

KV II

Recall : divergence in commutative calculus

$$u = u_x \partial_x + u_y \partial_y$$

$$\mathrm{div}(u) = \partial_x u_x + \partial_y u_y$$

Fact : $\mathrm{div}([u,v]_{Lie}) =$

$$= u(\mathrm{div}(v)) - v(\mathrm{div}(u))$$

$$\mathrm{div} : \mathfrak{X}(\mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^2)$$

is a Lie algebra 1-cocycle

$j(\varphi) := \log J(\varphi)$ log-Jacobian

- $j(\varphi \circ \psi) = j(\varphi) \circ \psi + j(\psi)$
- $\frac{d}{dt} j(\exp(tu)) \Big|_{t=0} = \mathrm{div}(u)$

Non-commutative divergence

$u \in \mathrm{Der}(\mathbb{L}(x,y))$

$u(x) = "\partial_x u(x)" \cdot x + "\partial_y u(x)" \, y$

$u(y) = \dots$

Theorem: $\mathrm{div} : \mathrm{Der}(\mathbb{L}) \to \mathrm{Cycl}(x,y)$

$$\mathrm{div}(u) = \mathrm{Cycl}(\partial_x u(x) + \partial_y u(y))$$

is a Lie algebra 1-cocycle

Example:

$u(x) = [x, y] \qquad \partial_x u(x) = -y$

$u(y) = [y, x] \qquad \partial_y u(y) = -x$

$\mathrm{div}(u) = -\mathrm{cycl}(x+y)$

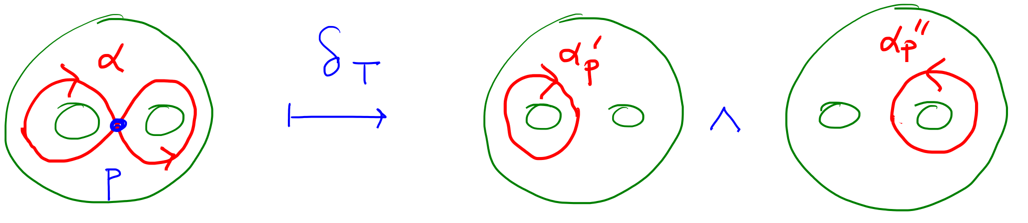
Remark: there is also a group cocycle $j : \mathrm{Aut}(\mathbb{L}) \to \mathrm{Cycl}(x,y)$

Theorem [Torossian, AA]

$KV\,\mathrm{II} \Leftarrow \quad j(\varphi) \in \mathfrak{z}\left(\mathrm{Cycl}(x,y), [\cdot,\cdot]_{gr}\right)$

Recall : Turaev cobracket

$$\delta_T : \mathfrak{g}(\Sigma) \longrightarrow \Lambda^2 \mathfrak{g}(\Sigma)$$



$$\delta_T(\alpha) = \sum_{p \in \alpha \cap \alpha} \varepsilon_p \, \alpha_p' \wedge \alpha_p''$$

[needs a choice of framing on Σ]

$\Rightarrow \quad \delta_{gr} : \mathrm{Cycl}(x,y) \longrightarrow \Lambda^2 \mathrm{Cycl}(x,y)$

[Schedler]

Theorem [AKKN]

Let $\varphi \in \mathrm{Aut}(\mathbb{L})$ solution of KVI.

Then,

(i) $\underline{\text{KVII}}$: $j(\varphi) \in \mathfrak{z}(\mathrm{Cycl}(x,y), [\cdot,\cdot]_{gr})$

$$\Updownarrow$$

(ii) $(\mathfrak{g}(\Sigma), [\cdot,\cdot]_G, \delta_T)$

$\cong \varphi$

$(\mathrm{Cycl}(x,y), [\cdot,\cdot]_{gr}, \delta_{gr})$

isomorphism of Lie bialgebras

Remark: KV problems for higher genus surfaces

$\swarrow$ $\searrow$?

moduli of flat connections

Representation Theory

Application: topological characterization of div

- $\mathrm{div}: \mathrm{Der}(\Bbbk\,\mathrm{Free}(x^{\pm 1}, y^{\pm 1})) \to \mathrm{Cycl}(x^{\pm 1}, y^{\pm 1})$

  depends on choice of generators

  dependence: $\in H_1(\Sigma) \cong \Bbbk x \oplus \Bbbk y$

Compare to: uniqueness of Haar measures

- Theorem [AKKN]

  For each framing $f : T\Sigma \xrightarrow{\cong} \Sigma \times \mathbb{R}^2$

  $\exists!\ \mathrm{div}^f$ s.t.

$$\delta^f_T = \mathrm{div}^f \circ \underbrace{[\cdot, -]_{\mathrm{Goldman}}}_{\in\ \mathrm{Der}(\Bbbk\,\mathrm{Free}(x^{\pm 1}, y^{\pm 1}))}$$

MANY HAPPY

RETURNS !